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The discrete Chazy III system of Labrunie–Conte is not integrable

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Abstract
We analyse the discrete form of the Chazy III equation proposed by Labrunie and Conte, with the help of two different integrability criteria: singularity confinement and algebraic entropy. We show that for all values of the free parameter this third-order mapping fails both criteria and thus cannot be integrable.

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The domain of integrable discrete systems has gone through a period of very fast growth in the past decade. The previous paucity of results has been replaced by a relative wealth where one can derive discrete integrable analogues of most well-known discrete systems. As far as one-dimensional mappings are concerned the most significant result is probably the discretization of the whole Painlevé/Gambier classification of integrable second-order differential equations [1] together with the derivation of the discrete analogues of the six transcendental Painlevé equations [2]. While the domain of integrable second-order mappings is well (although not yet fully) explored, the situation is different for higher-order mappings. The results on integrable third- or higher-order mappings are scant. In [3] we have presented a method for the systematic construction of integrable third-order mappings, which however are partially linearizable. Hirota and collaborators have presented some results on difference equations of third order which possess two conserved quantities [4]. Again in [3] we have shown that the so-called third-order q-P$_{\mathrm{I}}$ equation [5] can indeed be integrated once to an, already known, second-order q-Painlevé equation. Thus, every new result on integrable higher-order mappings would be most useful information at this stage.

In this paper, we shall examine the discretization of the Chazy III equation proposed in [6]. The authors started from the third-order ordinary differential equation:

$$u''' - 2u''u + 3u'^3 = 0 \tag{1}$$

and introduced the discretization:

$$
\begin{aligned}
u_{n+1} - 3u_n + 3u_{n-1} - u_{n-2} + 16(u_{n+1}u_{n-1} + u_n u_{n-2}) - 3u_n u_{n-1} - 27u_{n+1}u_{n-2} - u_{n+1}u_n \\
- u_{n-1}u_{n-2} + k(4(u_{n+1}u_{n-1} + u_n u_{n-2}) - 3u_n u_{n-1} - 3u_{n+1}u_{n-2} - u_{n+1}u_n \\
- u_{n-1}u_{n-2}) + m\left(u_{n+1}u_{n-2} + 4\left(u_n^2 + u_{n-1}^2\right)\right. \\
\left. - 7u_n u_{n-1} - u_{n+1}u_n - u_{n-1}u_{n-2}\right) = 0.
\end{aligned}
\tag{2}
$$

They then applied the perturbative Painlevé criterion [7] and concluded: 'Thus, there are great chances that equation (4) has the Painlevé property when $\mu_2' = 0$' (this being our equation (2) with $m = 0$). They went on to claim that 'the condition $\mu_2' = 0$ is also the only one given by the singularity confinement criterion'. (As we will show in what follows, this statement is not correct.) Now the (perturbative) Painlevé property and singularity confinement are two necessary integrability criteria [8], and thus one would expect the discrete Chazy III equation, satisfying both, to be a serious candidate for integrability. (Integrability is to be understood here as either the existence of a sufficient number of conserved quantities or the existence of a Lax pair leading to a linearization through spectral methods.) To be fair, the authors of [6] do not claim that (2) with $m = 0$ is integrable, although they speculate on its integration in their conclusions. As we shall show in what follows, mapping (2) is a typical nonintegrable discrete system. It does *not* possess the singularity confinement property and the degree growth of the iterates corresponds to a positive algebraic entropy (in the sense of [9]).

Let us first analyse (2) from the point of view of singularity confinement [10]. The keyword here is *singularity*. By this we mean any instance where the mapping loses one (or more) degrees of freedom, i.e. when an iterate does not depend on *all* the data introduced through the initial conditions. This is the notion of singularity introduced already in the very first works on the subject by the present authors [11, 12]. In some cases this loss of degree of freedom happens when an iterate assumes an infinite value, but this is not, by far, the only possible singularity (and it may not even be a singularity at all). Confinement in this context means the recovery of the lost degree of freedom and the way this can be done is through some indeterminate form such as $0/0$, $0 \times \infty$, $\infty - \infty$, etc.

Having set the frame we can now proceed to examine mapping (2) with $m = 0$. For a generic value of k we find that u_n may indeed pass through an infinite value at some iteration but u_{n+1} is finite and depends on the proper number of initial data. Thus this is precisely a case where ∞ is not a singularity at all. However, there also exist some nongeneric values of k where ∞ is indeed a singularity. For $k = -1$ we have the pattern $\{\infty, \infty\}$, i.e. if u_n is infinite (but u_{n-1} is not) then u_{n+1} is also infinite, but then u_{n+2} is finite and recovers the lost degree of freedom (through a $0/0$ indeterminacy) and this singularity is confined. For $k = -4$ we have the pattern $\{\infty, f, \infty\}$ where f is some finite value. In this case too the mapping recovers its full freedom after going through the second ∞. Finally, there exists another interesting value $k = -9$. Here one possible singularity pattern is $\{\infty, f, g, \infty\}$ where f, g are two finite values, with confined singularity. There is, however, also a possibility $\{\ldots\infty, f, g, \infty, h, n, \infty, p, q, \infty\ldots\}$ with the basic pattern $\{\infty, r, s\}$ (each letter representing some finite value) repeating *ad infinitum*. This is *not*, however, a case of unconfined singularity since it extends all the way to infinity in *both* directions.

However, the value $u = \infty$ is not the only singularity of mapping (2). Going back to the definition we gave above, it is clear that a singularity appears, *for some n*, when u_{n+1} does not depend on u_{n-2}. We solve for u_{n+1} from (2):

$$
u_{n+1} = \frac{4(k+4)u_{n-2}u_n - ((k+1)u_{n-1} + 1)(u_{n-2} + 3u_n) + 3u_{n-1}}{(k+1)u_n - 4(k+4)u_{n-1} + 3(k+9)u_{n-2} - 1}. \tag{3}
$$

Requiring $\partial u_{n+1}/\partial u_{n-2} = 0$ we find the following equation for u_n, u_{n-1}:

$$(k+1)(k+4)\left(u_n^2 - u_{n-1}^2\right) + 36u_n u_{n-1} + (k+16)(u_{n-1} - u_n) - 1/4 = 0. \tag{4}$$

We remark that when k takes one of the special values $k = -1$ or $k = -4$, the relation between u_n and u_{n-1} is of the first degree separately in each variable. This has no special significance, as far as singularity confinement is concerned, though it certainly does simplify the computations. The confinement of this singularity (which, let us point out, was ignored by the authors of [6]) means that u must recover the lost degree of freedom through some indeterminate form 0/0. We have checked that this *does not happen*. Indeed, iterating the mapping for a generic k and some 20 iterates, we did not encounter a 0/0 form. In the cases $k = -1, k = -4$, we were able to push the calculations to more than 100 iterations. Again, no possibility of confinement arose. Thus (and given the particularly large number of iterations) we can conclude with some confidence that mapping (2) does *not* possess the singularity confinement property. This is a first indication that this mapping is not integrable.

We now turn to a second discrete integrability criterion of a rational mapping, which is based on the degree growth of the iterates of some initial condition. A quantity that can be most easily computed is the degree of the numerators or denominators of (the irreducible forms of) the iterates. This computation can be performed by introducing homogeneous coordinates and computing the homogeneity degree. The seminal ideas for this approach are due to Arnold [13] and Veselov [14]. The second author remarked epigrammatically that 'integrability has an essential correlation with the weak growth of certain characteristics'. The notion of weak growth was made quantitative by Viallet and collaborators [9, 15], leading to the introduction of algebraic entropy. The latter is defined as $E = \lim_{n\to\infty}(\log d_n)/n$, where d_n is the degree of the nth iterate. A generic, nonintegrable, mapping leads to an exponential growth of the degrees of the iterates and thus has a nonzero algebraic entropy, while an integrable mapping has a zero algebraic entropy. The reason why the degree growth of an integrable mapping is not maximal lies in the fact that during the successive iterations, the same polynomial factors appear in the numerator and the denominator of the fraction that represents the nth iterate of some initial condition and thus cancel out. This factor cancellation is at the origin of the singularity confinement: if during the iterations the dependent variable takes a value which corresponds to a root of a polynomial factor, this may lead to a singularity, which, however, will eventually disappear when the appropriate polynomial factor is cancelled out.

For the initial condition $u_0, u_1, u_2 = p/q$, we have studied the degree growth in p, q of u_n as computed from (3). In the generic k case we have found the following sequence of degrees:

$$(0, 0, 1), 1, 2, 4, 7, 13, 24, 81, 149, 274, 504, 927, \ldots.$$

We are clearly in the presence of an exponential growth of the degrees. In fact as is expected, since the single singularity is nonconfined, no simplification ever occurs. Thus the degrees have the maximal possible growth. Since the numerator and denominator of u_{n+1} are of degree one separately in u_n, u_{n-1} and u_{n-2}, we have

$$d_{n+1} = d_n + d_{n-1} + d_{n-2}. \tag{5}$$

It is straightforward to compute the algebraic entropy of mapping (2). From the largest root of the equation $k^3 - k^2 - k - 1 = 0$, we obtain

$$E = \log\left(\frac{1}{3} + \left(\frac{19}{27} + \sqrt{\frac{11}{27}}\right)^{1/3} + \left(\frac{19}{27} - \sqrt{\frac{11}{27}}\right)^{1/3}\right)$$

with a numerical value of $\log(1.84\ldots) \approx 0.61$ which is in agreement with the calculated ratio of the degrees of the iterates.

We now turn to the nongeneric cases $k = -1, -4$. In these cases, since there is a second, confined, singularity in addition to the nonconfined one, we expect *some* simplification to occur and thus the growth should be slower than in the generic case. For $k = -1$ we find indeed the degree sequence

$$(0, 0, 1), 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, \ldots.$$

This is again an exponential growth. The relation between the degrees is now

$$d_{n+1} = d_{n-1} + d_{n-2} + d_{n-3} \tag{5}$$

and thus the algebraic entropy is obtained by the largest root of the equation $k^4 - k^2 - k - 1 = 0$:

$$E = -\log\left(\left(\sqrt{\frac{31}{108}} + \frac{1}{2}\right)^{1/3} - \left(\sqrt{\frac{31}{108}} - \frac{1}{2}\right)^{1/3}\right)$$

leading to a numerical value of $\log(1.46\ldots) \approx 0.38$. In the case $k = -4$, we have

$$(0, 0, 1), 1, 2, 3, 4, 7, 11, 17, 27, 42, 66, 104, \ldots.$$

The recursion of the degrees is now

$$d_{n+1} = d_n + d_{n-2} + d_{n-4}. \tag{6}$$

The asymptotic ratio of two successive degrees is given by the single real root of the equation $k^5 = k^4 + k^2 + 1$, which has the approximate value of $k = 1.57\ldots$ leading to an algebraic entropy of 0.45. Thus the special cases $k = -1, -4$, despite some simplifications, still lead to exponential degree growth and thus are expected to be nonintegrable.

The main result of this paper is the demise of the Labrunie–Conte discretization of the Chazy III equation. Thus the question of the existence of an integrable Chazy III mapping is still open. We would like to conclude this paper with a remark concerning the application of singularity confinement. In the case of continuous singularity analysis, one must find *all* leading singular behaviours and examine the expansions around them before being able to make a statement on the Painlevé property of the system. Similarly, in the case of discrete systems the proper application of the singularity confinement criterion requires the examination of *all* possible singularities. As we explained above, a singularity is a loss of a degree of freedom (and not just a diverging value of some iterate). Neglecting these singularities can lead to erroneous conclusions. In the case at hand the *proper* analysis of the Chazy III mapping of [6] allowed us to determine its nonintegrable character in an unambiguous way.

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